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ON THE IRREDUCIBILITY OF THE EQUATIONS OF THE RESTRICTED CIRCULAR, THREE-BODY PROBLEM TO THE STÄCKEL TYPE EQUATIONS

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It is shown that no generalized coordinates exist in which the total integral of the Hamilton-Jacobi equation for a restricted circular, three-body problem can be represented in the form of a finite sum of the functions each of which depends on a single generalized coordinate.

The Hamilton-Jacobi equation for a restricted plane circular, three-body problem in elliptical variables u, v has the form [1]

$$\left(\frac{\partial S}{\partial u}\right)^2 + \left(\frac{\partial S}{\partial v}\right)^2 + \frac{\partial S}{\partial u}\left[nc^2 \sin 2v - nc\,(a_1 - a_0)\,\mathrm{ch}\, u \sin v\right] + \frac{\partial S}{\partial v}\left[nc^2\,\mathrm{sh}\, 2u - nc\,(a_1 - a_0)\,\mathrm{sh}\, u \cos v\right] = F_1(u) + F_2(v) \qquad (1)$$

$$F_1(u) = hc^2 \operatorname{ch} 2u + 2fc\,(m_0 + m_1) \operatorname{ch} u$$
$$F_2(v) = -hc^2 \cos 2v + 2fc\,(m_0 - m_1) \cos v$$

Here f is the gravitational constant, h is the constant of the Jacobi integral, n is the angular velocity of rotation of the attracting points P_0 and P_1 of mass m_0 and m_1, respectively, about a common center of mass G. Other parameters are shown on Fig. 1 where

$$r_0^2 = (x + c)^2 + y^2, \quad r_1^2 = (x - c)^2 + y^2$$
$$|P_0G| = a_0, \quad |P_1G| = a_1$$

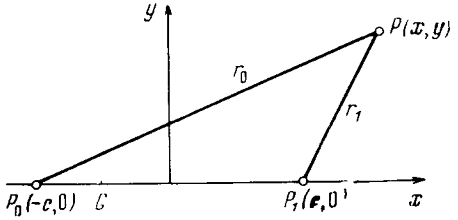

Fig. 1

The elliptical and rectangular coordinates of the point P are connected by the following relations:

$$2c \operatorname{ch} u = r_0 + r_1, \qquad 2c \cos v = r_0 - r_1$$

In the case of two independent variables, we shall call the nonlinear partial differential equation

$$f_1(q_1)\left[\frac{\partial W}{\partial q_1} + \varphi_1(q_1)\right]^2 + f_2(q_2)\left[\frac{\partial W}{\partial q_2} + \varphi_2(q_2)\right]^2 = \Phi_1(q_1) + \Phi_2(q_2) \qquad (2)$$

the Stäckel equation. This is the form which the Hamilton-Jacobi equation assumes when it is integrated by the method of separation of variables. As we know [2, 4] the Stäckel and Liouville systems and their generalizations are reduced to this form.

The Stäckel theorem [2, 5] gives the necessary and sufficient conditions for the separation of variables in the Hamilton-Jacobi equation, i. e. it contains the conditions under which the Hamilton-Jacobi equation written in the independent variables q_1 and q_2 is reduced to the form (2) in the same variables.

Let us perform the following differentiable nondegenerate change of variables

$$u = \psi_1(q_1, q_2), \qquad v = \psi_2(q_1, q_2) \qquad (3)$$

denoting the inverse operation by

$$q_1 = \omega_1(u, v), \qquad q_2 = \omega_2(u, v)$$

Theorem. No nondegenerate differentiable change of variables of the type (3) exists which transforms the Hamilton-Jacobi equation (1) into the Stäckel equation (2).

Proof. The conditions of transformation of (1) into (2) contain

$$\left(\frac{\partial \omega_i}{\partial u}\right)^2 + \left(\frac{\partial \omega_i}{\partial v}\right)^2 = f_i(\omega_i) \qquad (i = 1, 2) \qquad (4)$$

$$\frac{\partial \omega_1}{\partial u}\frac{\partial \omega_2}{\partial u} + \frac{\partial \omega_1}{\partial v}\frac{\partial \omega_2}{\partial v} = 0$$

$$F_1(u) + F_2(v) = \Phi_1(\omega_1) + \Phi_2(\omega_2) - f_1(\omega_1)\varphi_1^2(\omega_1) - f_2(\omega_2)\varphi_2^2(\omega_2)$$

$$[nc^2 \sin 2v - nc\,(a_1 - a_0) \operatorname{ch} u \sin v]\frac{\partial \omega_i}{\partial u} +$$

$$[nc^2 \operatorname{sh} 2u - nc\,(a_1 - a_0) \operatorname{sh} u \cos v]\frac{\partial \omega_i}{\partial v} = 2f_i(\omega_i)\varphi_i(\omega_i) \qquad (i = 1, 2)$$

The system (4) consists of three nonlinear partial differential equations, two quasi-linear and one functional equation, and has eight unknown functions $(\omega_1, \omega_2, f_1, f_2, \varphi_1, \varphi_2, \Phi_1, \Phi_2)$. We shall prove that if $n = 0$ and $c \neq 0$, then the system (4) is incompatible no matter what functions are chosen to appear in it.

The total integrals of the first and second equations of (4) are obtained using the Lagrange-Charpy method [6]

$$u + C_1 v + C_2 = \sqrt{1 + C_1^2} \int \frac{d\omega_1}{\sqrt{f_1(\omega_1)}} \tag{5}$$

$$C_3 u + v + C_4 = \sqrt{1 + C_3^2} \int \frac{d\omega_2}{\sqrt{f_2(\omega_2)}}$$

where C_i are arbitrary constants. The condition of nondegeneracy of (5) is $C_1 C_3 - 1 \neq 0$. Taking (5) into account we can prove that the third equation of (4) is satisfied when $C_1 = -C_3$, therefore the substitution

$$u = \frac{1}{\sqrt{1 + C_1^2}} \left[\int \frac{dq_1}{\sqrt{f_1(q_1)}} - C_1 \int \frac{dq_2}{\sqrt{f_2(q_2)}} \right] + \frac{C_1 C_4 - C_2}{1 + C_1^2} \tag{6}$$

$$v = \frac{1}{\sqrt{1 + C_1^2}} \left[C_1 \int \frac{dq_1}{\sqrt{f_1(q_1)}} + \int \frac{dq_2}{\sqrt{f_2(q_2)}} \right] - \frac{C_1 C_2 + C_4}{1 + C_1^2}$$

with any positive functions $f_1(q_1)$ and $f_2(q_2)$, satisfies the first three equations of (4).

Let us now consider the fourth eqaution of (4). Substituting the relations (6) into the expressions for $F_1(u)$ and $F_2(v)$ and performing the necessary manipulations we can conclude, that the fourth equation of (4) holds when and only when $C_1 = 0$, since only in this case $F_1(u) + F_2(v)$ remains a function with separated variables after the variable change. This means that

$$u = \psi_1(q_1), \quad v = \psi_2(q_2), \quad q_1 = \omega_1(u), \quad q_2 = \omega_2(v)$$

But the fifth equation of (4) has no solutions of the type $q_1 = \omega_1(u)$ irrespective of the choice of the functions $f_1(\omega_1)$ and $\varphi_1(\omega_1)$ (similarly the sixth equation has no solutions of the type $q_2 = \omega_2(v)$).

Thus no curvilinear coordinates exist which could be used to transform the Hamilton-Jacobi equation for a plane restricted.three-body problem into a Stäckel type equation (2).

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